Entropy of some automorphisms of the II_1-factor of the free group in infinite number of generators

by

Erling Størmer
Matematisk institutt
Universitetet i Oslo
Blindern, 0316 Oslo, Norway

Abstract

Let $L(\mathbf{F}_\infty)$ be the II_1-factor defined by the free group $\mathbf{F}_\infty$ in infinite number of generators. It is shown that for a class of automorphisms of $L(\mathbf{F}_\infty)$ arizing from bijections of the set of generators of $\mathbf{F}_\infty$ on itself, and including the free shift, the entropy is zero.

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1 Introduction

In order to obtain a deeper understanding of noncommutative entropy as defined by Connes and co-authors [2,3,4] it is necessary to compute the entropy in a variety of cases. In the present paper we shall consider the II_1-factor $L(\mathbf{F}_\infty)$ defined by the left regular representation of the free group $\mathbf{F}_\infty$ in countably infinite number of generators. If G is the set of generators then each bijection $\alpha : G \to G$ defines an automorphism of $\mathbf{F}_\infty$ and hence of $L(\mathbf{F}_\infty)$. We shall prove that if each orbit $\{\alpha^n(u) : n \in \mathbf{Z}\}$, $u \in G$, is infinite then the entropy of α is zero. A special example is the free shift, which if G is indexed by $\mathbf{Z}$ corresponds to the shift on $\mathbf{Z}$. This automorphism is extremely ergodic in the sense that the only globally invariant injective von Neumann subalgebra of $L(\mathbf{F}_\infty)$ is the scalars [6]. Now it has been noted by Voiculescu, see e.g. [7], that free products of C^*-algebras have many features analogous to infinite tensor products. Thus it may seem surprising that the free shift has entropy zero. The reason is that $L(\mathbf{F}_\infty)$ is highly noncommutative with a great deal of dependence between factors of free products of subalgebras, so that operators tend to be orthogonal in the Hilbert space structure defined by the trace. The main content of section 2 is a uniform estimate for finite sets of positive operators to this effect. This is quite different from infinite tensor products, which have lots of independence and are close to the classical abelian situation. In particular the shift has a globally invariant subalgebra on which the action is the classical one.

In two recent papers [8,9] Voiculescu studied another concept of entropy of automorphisms of von Neumann algebras, which he called perturbation theoretic entropy. He showed that for the free shift this entropy has the value $+\infty$, thus the two types of entropy are essentially different in $L(\mathbf{F}_\infty)$.

2 Norm estimates on $L(\mathbf{F}_\infty)$

Let M be a II_1-factor with normalized trace τ. Let $N \subset M$ be a von Neumann subalgebra. We denote by E_M the trace preserving conditional expectation of M onto N defined by the equation

$$\tau(E_N(x)y) = \tau(xy) \quad \text{for } x \in M,\ y \in N\,.$$

The L^1-norm of an element $x \in M$ is given by

$$\|x\|_1 = \tau(|x|) = \sup_{y \in M_1} |\tau(xy)|\,,$$

where M_1 denotes the unit ball in M, see [5, Ch. 1, §6, Cor. 1]. Thus we have

$$\|E_N(x)\|_1 = \sup_{y \in N_1} |\tau(E_N(x)y)| = \sup_{y \in N_1} |\tau(xy)| \le \sup_{y \in M_1} |\tau(xy)| = \|x\|_1\,.$$

Thus in obvious notation $\|E_N\|_1 \le 1$. Let E_N^0 denote the orthogonal projection in $L^2(M,\tau)$ onto the orthogonal complement of 1 in N, i.e.

$$E_N^0(x) = E_N(x) - \tau(x)1\,.$$

Since $|\tau(x)| \le \|x\|_1$ we have

$$\|E_N^0(x)\|_1 \le \|E_N(x)\|_1 + |\tau(x)| \le 2\|x\|_1\,,$$

hence

$$\|E_N^0\|_1 \le 2\,.$$

We shall in the rest of this section study the following situation.

2.1 Notation Let $(S_j)_{j \in \mathbf{N}}$ be a sequence of disjoint subsets of the set of generators G of $\mathbf{F}_\infty$. Let $\mathbf{F}_j = \mathbf{F}_{S_j}$ be the free subgroup of $\mathbf{F}_\infty$ generated by S_j, and let $\mathbf{F}_j^0 = \mathbf{F}_j - \{e\}$, where e is the identity in $\mathbf{F}_\infty$. Let $E_j = E_{L(\mathbf{F}_j)}$ and $E_j^0 = E_{L(\mathbf{F}_j)}^0$. Each element $g \in \mathbf{F}_\infty$, $g \ne e$, can be written as a product $u_1^{n_1} u_2^{n_2} \ldots u_k^{n_k}$, where $u_1, \ldots, u_k \in G$, $n_j \in \mathbf{Z} - \{0\}$, and $u_i \ne u_{i+1}$. Then g is said to be in reduced form. We denote by J_j the set of $g \in \mathbf{F}_\infty$ such that g does not end to the right in S_j, i.e. either $g = e$ or if $g = u_1^{n_1} \ldots u_k^{n_k}$ as above, then $u_k \notin S_j$.

We denote by $g \to \lambda_g$ the left regular representation of $\mathbf{F}_\infty$ into $L(\mathbf{F}_\infty)$.

Lemma 2.2. Let notation be as above. Let $x = \sum_{g \in \mathbf{F}_\infty} a_g \lambda_g$, $a_g \in \mathbf{C}$, be a finite sum. Then

$$E_j^0(x^*x) = \sum_{h \in J_j} \left(\Big| \sum_{t \in \mathbf{F}_j} a_{ht} \lambda_{ht} \Big|^2 - \sum_{t \in \mathbf{F}_j} |a_{ht}|^2 \right)$$

Proof: We have $x^*x = \sum\limits_{g,h\in\mathbf{F}_\infty} \bar{a}_g a_h \lambda_{g^{-1}h}$. Since $g^{-1}h \in \mathbf{F}_j$ if and only if $h = gs$ with $s \in \mathbf{F}_j$, hence, when g is written in the form $g = g_0 t$, $g_0 \in J_j$, $t \in \mathbf{F}_j$, then $g^{-1}h \in \mathbf{F}_j$ if and only if $g = g_0 t$, $h = g_0 ts$, $g_0 \in J_j$, $t, s \in \mathbf{F}_j$. It follows that the elements in the sum of x^*x with $g^{-1}h \in \mathbf{F}_j^0$ are those of the form

$$\bar{a}_{ht} a_{hts} \lambda_s\,, \qquad h\in J_j,\ t\in\mathbf{F}_j\,,\ s\in\mathbf{F}_j^0\,.$$

Thus we have

$$x^*x = \sum_{g\in\mathbf{F}_\infty} |a_g|^2 + \sum_j \sum_{s\in\mathbf{F}_j^0} \left(\sum_{h\in J_j} \sum_{t\in\mathbf{F}_j} \bar{a}_{ht} a_{hts} \right) \lambda_s + c\,, \tag{1}$$

where c is an element in $L(\mathbf{F}_\infty)$ which is orthogonal to all subalgebras $L(\mathbf{F}_j)$. We have as above

$$\begin{aligned} \left| \sum_{t\in\mathbf{F}_j} a_{ht}\lambda_{ht} \right|^2 &= \sum_{s\in\mathbf{F}_j} \left(\sum_{t\in\mathbf{F}_j} \bar{a}_{ht} a_{hts} \right) \lambda_s \\ &= \sum_{t\in\mathbf{F}_j} |a_{ht}|^2 + \sum_{s\in\mathbf{F}_j^0} \left(\sum_{t\in\mathbf{F}_j} \bar{a}_{ht} a_{hts} \right) \lambda_s\,. \end{aligned}$$

Substitution of this in (1) yields

$$x^*x = \sum_{g\in\mathbf{F}_\infty} |a_g|^2 + \sum_j \sum_{h\in J_j} \left(\left| \sum_{t\in\mathbf{F}_j} a_{ht}\lambda_{ht} \right|^2 - \sum_{t\in\mathbf{F}_j} \left| a_{ht} \right|^2 \right) + c\,,$$

whence

$$E_j^0(x^*x) = \sum_{h\in J_j} \left(\left| \sum_{t\in\mathbf{F}_j} a_{ht}\lambda_{ht} \right|^2 - \sum_{t\in\mathbf{F}_j} \left| a_{ht} \right|^2 \right).$$

QED.

Lemma 2.3. Let $\varepsilon > 0$. Then there exists $r = r(\varepsilon) \in \mathbf{N}$ with the following property: If $(x^\ell)_{\ell\in I}$ is a finite set of operators in $L(\mathbf{F}_\infty)^+$ of the form x^*x in Lemma 2.2, then there exists a subset $J \subset \mathbf{N}$ with card $J \leq r$ such that

$$\sum_{\ell\in I} \|E_j^0(x^\ell)\|_1 < \varepsilon\tau\Big(\sum_{\ell\in I} x^\ell\Big) \quad \text{for } j \notin J\,.$$

Proof Normalizing we may assume $\tau\Big(\sum\limits_{\ell\in I} x^\ell\Big) = 1$. We have by assumption $x^\ell = y_\ell^* y_\ell$ with $y_\ell = \sum\limits_{g\in\mathbf{F}_\infty} a_g^\ell \lambda_g$. Thus

$$\sum_{\ell\in I}\sum_{g\in\mathbf{F}_\infty}|a_g^\ell|^2=\sum_\ell\|y^\ell\|_2^2=\sum_\ell\|x^\ell\|_1=\sum_\ell\tau(x^\ell)=1\,.$$

From formula (1) we have

$$x^\ell=\sum_{g\in\mathbf{F}_\infty}|a_g^\ell|^2+\sum_j\sum_{s\in\mathbf{F}_j^0}\sum_{h\in J_j}\sum_{t\in\mathbf{F}_j}\bar{a}_{ht}^\ell a_{hts}^\ell\lambda_s+c^\ell\,,$$

with $c^\ell\perp L(\mathbf{F}_j)$. Note that the term $|a_{ht}^\ell|^2$ appears only once in the sum

$$\sum_j\sum_{\ell\in I}\sum_{h\in J_j}\sum_{t\in\mathbf{F}_j^0}|a_{ht}^\ell|^2\,.$$

Thus this sum has value majorized by $\sum_{\ell\in I}\sum_{g\in\mathbf{F}_\infty}|a_g^\ell|^2=1\,.$

Let $\delta>0$ satisfy $2(\delta^{1/2}+\delta)=\varepsilon$, and let $r=\left[\frac{1}{\delta}\right]+1$. If $c_i\geq 0$ and $\sum_{i=1}^\infty c_i=1$ we have $c_i\geq\delta$ for at most r indicies i. Thus there exists a subset $J\subset\mathbf{N}$ with card $J\leq r$ such that

$$\sum_{\ell\in I}\sum_{h\in J_j}\sum_{t\in\mathbf{F}_j^0}|a_{ht}^\ell|^2<\delta\qquad\text{for } j\notin J\,. \tag{2}$$

By Lemma 2.2

$$E_j^0(x^\ell)=\sum_{h\in J_j}\left(\left|\sum_{t\in\mathbf{F}_j}a_{ht}^\ell\lambda_{ht}\right|^2-|a_h^\ell|^2-\sum_{t\in\mathbf{F}_j^0}|a_{ht}^\ell|^2\right). \tag{3}$$

Let $y_h^\ell=\sum_{t\in\mathbf{F}_j^0}a_{ht}^\ell\lambda_t$. Since $\sum_{t\in\mathbf{F}_j}a_{ht}^\ell\lambda_{ht}=\lambda_h\sum_{t\in\mathbf{F}_j}a_{ht}^\ell\lambda_t$ we have

$$\left|\sum_{t\in\mathbf{F}_j}a_{ht}^\ell\lambda_{ht}\right|^2=|a_h^\ell+y_h^\ell|^2=|a_h^\ell|^2+\bar{a}_h^\ell y_h^\ell+a_h^\ell y_h^{\ell*}+y_h^{\ell*}y_h^\ell\,.$$

Since for $x\in L(\mathbf{F}_\infty)$, $\tau(|x|)^2\leq\tau(|x|^2)$, hence $\|x\|_1\leq\|x\|_2$, and furthermore

$$\sum_{h\in J_j}\sum_{\ell\in I}\|y_h^\ell\|_2^2=\sum_{h\in J_j}\sum_{\ell\in I}\sum_{t\in\mathbf{F}_j^0}|a_{ht}^\ell|^2<\delta\qquad\text{for } j\notin J\,,$$

we have by (2) and (3) for $j\notin J$

$$\begin{aligned}
\sum_{\ell\in I}\|E_j^0x^\ell\|_1 &\leq \sum_{\ell\in I}\sum_{h\in J_j}\left\|\bar{a}_h^\ell y_h^\ell+a_h^\ell y_h^{\ell*}+y_h^{\ell*}y_h^\ell-\sum_{t\in\mathbf{F}_j^0}|a_{ht}^\ell|^2\right\|_1\\
&\leq \sum_{\ell\in I}\sum_{h\in J_j}\left(2|a_h^\ell|\,\|y_h^\ell\|_1+\|y_h^\ell\|_2^2+\sum_{t\in\mathbf{F}_j^0}|a_{ht}^\ell|^2\right)\\
&\leq 2\sum_{\ell\in I}\sum_{h\in J_j}|a_h^\ell|\,\|y_h^\ell\|_2+2\sum_{\ell\in I}\sum_{h\in J_j}\sum_{t\in\mathbf{F}_j^0}|a_{ht}^\ell|^2\\
&< 2\Big(\sum_{\ell\in I}\sum_{h\in J_j}|a_h^\ell|^2\Big)^{1/2}\Big(\sum_{\ell\in I}\sum_{h\in J_j}\|y_h^\ell\|_2^2\Big)^{1/2}+2\delta\\
&\leq 2\delta^{1/2}+2\delta=\varepsilon\,.
\end{aligned}$$

QED.

Lemma 2.4. Let $\varepsilon > 0$. Then there exists $r = r(\varepsilon) \in \mathbf{N}$ with the following property: Let notation be as in (2.1) and let $(x^\ell)_{\ell \in I}$ be a finite subset of $L(\mathbf{F}_\infty)^+$. Then there exists a subset $J \subset \mathbf{N}$ with card $J \leq r$ such that

$$\sum_{\ell \in I} \|E_j^0(x^\ell)\|_1 < \varepsilon\tau\Big(\sum_{\ell \in I} x^\ell\Big) \qquad \text{for } j \notin J\,.$$

Proof. Normalizing we may assume $\sum_{\ell \in I} \tau(x^\ell) = 1$. Let $\mathcal{A}$ denote the $*$-algebra consisting of finite sums $\sum_{g \in \mathbf{F}_\infty} a_g \lambda_g$, $a_g \in \mathbf{C}$. Then $\mathcal{A}$ is strongly dense in $L(\mathbf{F}_\infty)$. Let $k = \max_{\ell \in I} \|x^\ell\|^{1/2}$. By the Kaplansky density theorem the ball $\mathcal{A}_k$ of operators with norm less than k is strongly dense in $L(\mathbf{F}_\infty)_k = \{x \in L(\mathbf{F}_\infty) : \|x\| \leq k\}$. Write $x^\ell = y^{\ell^*} y^\ell$, with $y \in L(\mathbf{F}_\infty)$. Then there is a net $(y_\alpha^\ell)_{\alpha \in A}$ in $\mathcal{A}_k$ which converges strongly to y^ℓ. Since multiplication is strongly continuous on bounded sets and $*$-operation is strongly continuous in II_1-factors, $y_\alpha^{\ell^*} y_\alpha^\ell \to x^\ell$ strongly. From the inequality

$$\|x - y\|_1^2 \leq \|x - y\|_2^2 = \tau(x^2 - 2xy + y^2) \qquad \text{for } x, y \in L(\mathbf{F}_\infty)^+$$

we can conclude that $\|y_\alpha^{\ell^*} y_\alpha^\ell - x^\ell\|_1 \to 0$. Let $s =$ card I, and choose $y^\ell \in \mathcal{A}_k$ with $\|x^\ell - y^{\ell^*} y^\ell\|_1 < \varepsilon/4s$, and such that $\tau(y^{\ell^*} y^\ell) = \tau(x^\ell)$. Then $\sum_{\ell \in I} \tau(y^{\ell^*} y^\ell) = 1$. Let $r = r(\varepsilon/2)$, with $r(\varepsilon/2)$ as in Lemma 2.3, and let $J \subset \mathbf{N}$ be a set with card $J \leq r$ such that

$$\sum_{\ell \in I} \|E_j^0(y^{\ell^*} y^\ell)\|_1 < \varepsilon/2 \qquad \text{for } j \notin J\,.$$

Since $\|E_j^0\|_1 \leq 2$ we thus have for $j \notin J$,

$$\begin{aligned}
\sum_{\ell \in I} \|E_j^0(x^\ell)\|_1 &\leq \sum_{\ell \in I} \|E_j^0(y^{\ell^*} y^\ell)\|_1 + \sum_{\ell \in I} \|E_j^0(y^{\ell^*} y^\ell - x^\ell)\|_1 \\
&< \varepsilon/2 + 2\sum_{\ell \in I} \|x^\ell - y^{\ell^*} y^\ell\|_1 \\
&< \varepsilon/2 + 2s\,\varepsilon/4s \\
&= \varepsilon\,.
\end{aligned}$$

QED.

We shall need a version of the last lemma in which the L^1-norm $\| \ \|_1$ is replaced by the operator norm $\| \ \|$.

Lemma 2.5. Let notation be as in (2.1). Let N_1 be a finite dimensional von Neumann subalgebra of $L(\mathbf{F}_1)$. Suppose for each j there is an automorphism α_j of $L(\mathbf{F}_\infty)$ such that $N_j = \alpha_j(N_1) \subset L(\mathbf{F}_j)$. Then given $\varepsilon > 0$ there exists $r = r(\varepsilon, N_1) \in \mathbf{N}$ such that if $(x^\ell)_{\ell\in I}$ is a finite subset of $L(\mathbf{F}_\infty)^+$, then there is a subset $J \subset \mathbf{N}$ with card $J \le r$ such that

$$\sum_{\ell\in I} \|E^0_{N_j}(x^\ell)\| < \varepsilon\tau\Big(\sum_{\ell\in I} x^\ell\Big) \qquad \text{for } j \notin J\,.$$

Proof. Since N_1 is finite dimensional there exists a constant $c > 0$ such that $\|x\| \le c\|x\|_1$ for $x \in N_1$. Since an automorphism of a II_1-factor is isometric for both norms $\|\ \|$ and $\|\ \|_1$ the same inequality holds for $x \in N_j$ for all j. With r as in Lemma 2.4 let $r(\varepsilon, N_1) = r(\varepsilon/c)$. Then for $x \in L(\mathbf{F}_\infty)^+$

$$\|E^0_{N_j}(x)\| \le c\|E^0_{N_j}(x)\|_1 \le c\|E^0_j(x)\|_1\,,$$

So the lemma follows from Lemma 2.4.

QED.

3 Entropy

In this section we prove our main result on entropy announced in the introduction. The automorphisms we shall consider are those which act freely on the generators of $\mathbf{F}_\infty$, where the definition of free will be as follows:

Definition 3.1. If X is a set and $\alpha : X \to X$ is a bijection we say α is *free* if the cyclic group $\{\alpha^n : n \in \mathbf{Z}\}$ acts freely, or equivalently, the map $n \to \alpha^n(x)$ of $\mathbf{Z}$ into X is injective for all $x \in X$.

The following result is probably well-known and is included for completeness. We are indebted to N. Øvrelid for showing it to us.

Lemma 3.2. Let X be a set and $\alpha : X \to X$ a bijection. Then the following three conditions are equivalent.

(i) α is free.

(ii) Each orbit $\{\alpha^n(x) : n \in \mathbf{Z}\}$, $x \in X$, is infinite.

(iii) For each finite subset $S \subset X$ there is $p \in \mathbf{N}$ such that the sets $\alpha^{np}(S)$, $n \in \mathbf{Z}$, are disjoint.

Proof. The equivalence (i)$\Leftrightarrow$(ii) is trivial. If $\alpha^n(x) = \alpha^m(x)$ with $n \neq m$, then $\alpha^{p(n-m)}(x) = x$ for all $p \in \mathbf{N}$, hence the sets $\alpha^{jp}(\{x\})$, $j \in \mathbf{Z}$, are not disjoint. Thus (iii)$\Rightarrow$(i).

Assume (i), and let $S = \{x_1, \ldots, x_r\}$ be a finite subset of X. Since orbits are disjoint or equal it suffices to show (iii) in the case when S is contained in an orbit. Say $x_i = \alpha^{n_i}(x)$, $x \in X$, $i = 1, \ldots, r$. Let $p = 1 + 2\max_i |n_i|$. Then the sets $\alpha^{np}(S)$, $n \in \mathbf{Z}$, are all disjoint, as follows easily from freeness of α.

QED.

To fix notation we recall some facts from [4]. τ is a normal trace of a finite von Neumann algebra M with $\tau(1) = 1$. For each $k \in \mathbf{N}$, S_k consists of the set of all families $(x_{i_1 \ldots i_k})_{i_j \in \mathrm{N}}$ of positive elements in M, zero except for a finite number of indices, and satisfying

$$\sum_{i_1, \ldots, i_k} x_{i_1 \ldots i_n} = 1\,.$$

We say such a family $(x_{i_1 \ldots i_n})$ is a partition of unity. For $x \in S_k$, $\ell \in \{1, \ldots, k\}$ and $i_\ell \in \mathrm{N}$ we put

$$x^\ell_{i_\ell} = \sum_{i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_k} x_{i_1 \ldots i_k}\,.$$

If $N_1, \ldots, N_k$ are finite dimensional von Neumann subalgebras of M, then

$$H(N_1, \ldots, N_k) = \sup_{x \in S_k} \sum_{i_1 \ldots i_k} \eta\tau(x_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta E_{N_\ell}(x^\ell_{i_\ell})\,.$$

Here $\eta(t)$ is the real function on $[0,1]$, $\eta(0) = 0$, $\eta(t) = -t \log t$. If $\alpha \in \operatorname{Aut} M$ and $N \subset M$ is finite dimensional then

$$H(N, \alpha) = \lim_{k \to \infty} \tfrac{1}{k} H(N, \alpha(N), \ldots, \alpha^{k-1}(N))\,.$$

The entropy $H(\alpha)$ is defined by

$$H(\alpha) = \sup_N H(N, \alpha)\,,$$

the sup taken over all finite dimensional von Neumann subalgebras.

As noted in the introduction a bijection of the set G of generators of $\mathbf{F}_\infty$ onto itself defines an automorphism of $L(\mathbf{F}_\infty)$.

Theorem 3.3. Let α be an automorphism of $L(\mathbf{F}_\infty)$ defined by a free bijection of the set of generators on itself. Then its entropy $H(\alpha) = 0$.

The theorem is probably true for all bijections of G on itself, since the entropy of a periodic automorphism is zero, and each bijection is a combination of free and periodic maps. Thus an automorphism on $L(\mathbf{F}_\infty)$ defined in this way is a "free product" of free and periodic actions. We thus get into a problem analogous to that of considering the entropy of a tensor product of automorphisms, a problem which is still unsolved.

By [4, Remark 6] the entropy satisfies the following identity for automorphisms of the hyperfinite II_1-factor.

$$H(\alpha^p) = |p| H(\alpha), \qquad p \in \mathbf{Z}.$$

We shall be content with a weaker result. We remark that the lemma can as easily be proved in the greater generality of [3].

Lemma 3.4. Let M be a finite von Neumann algebra and $\alpha \in \mathrm{Aut}(M)$ with $\tau \circ \alpha = \tau$. Let $p \in \mathbf{N}$. Then for $n = pk$, $k \in \mathbf{N}$ we have for a finite dimensional von Neumann subalgebra $N \subset M$

$$\tfrac{1}{n} H(N, \alpha(N), \ldots, \alpha^{n-1}(N)) \leq \tfrac{1}{k} H(N, \alpha^p(N), \ldots, \alpha^{p(k-1)}(N))$$

Proof. By subadditivity of H we have

$$\begin{aligned} H(N, \alpha(N), \ldots, \alpha^{n-1}(N)) &\leq H(N, \alpha^p(N), \ldots, \alpha^{p(k-1)}(N)) \\ &+ H(\alpha(N), \alpha^{p+1}(N), \ldots, \alpha^{p(k-1)}\alpha(N)) \\ &+ \cdots + H(\alpha^{p-1}(N), \alpha^{2p-1}(N), \ldots, \alpha^{p(k-1)}\alpha^{p-1}(N)) \\ &= pH(N, \alpha^p(N), \ldots, \alpha^{p(k-1)}(N)). \end{aligned}$$

Hence

$$\begin{aligned} \tfrac{1}{n} H(N, \alpha(N), \ldots, \alpha^{n-1}(N)) &\leq \tfrac{p}{n} H(N, \alpha^p(N), \ldots, \alpha^{p(k-1)}(N)) \\ &= \tfrac{1}{k} H(N, \alpha^p(N), \ldots, \alpha^{p(k-1)}(N)) \end{aligned}$$

QED.

Lemma 3.5. Let M be a von Neumann algebra with a normal trace τ with $\tau(1) = 1$. Let $(x_{i_1 \ldots i_k}) \in S_k$ be a partition of unity in M^+. Then

$$\sum_{i_1 \ldots i_k} \eta\tau(x_{i_1 \ldots i_k}) - \sum_{\ell=1}^{k} \sum_{i_\ell} \eta\tau(x^{\ell}_{i_\ell}) \leq 0.$$

Proof. Let $N_\ell = \mathbf{C}$. Then $E_{N_\ell} = \tau$, so by definition of $H(N_1, \ldots, N_k)$ we get

$$\begin{aligned} 0 &= H(\mathbf{C}) = H(N_1, \ldots, N_k) \\ &= \sup_{y \in S_k} \sum_{i_1 \ldots i_k} \eta\tau(y_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta(\tau(y^\ell_{i_\ell})) \\ &= \sup_{y \in S_k} \sum_{i_1 \ldots i_k} \eta\tau(y_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \eta\tau(y^\ell_{i_\ell}) \\ &\geq \sum_{i_1 \ldots i_k} \eta\tau(x_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \eta\tau(x^\ell_{i_\ell}) . \end{aligned}$$

QED.

Lemma 3.6. Let $c_i \geq 0$, $\lambda_i \in [0,1]$, $i = 1, \ldots, s$, $\sum_{i=1}^{s} c_i = 1$. Let

$$\varepsilon = \max_i \left| \lambda_i - \sum_{j=1}^{s} c_j \lambda_j \right|$$

Then

$$0 \leq \eta\Big(\sum_{i=1}^{s} c_i \lambda_i \Big) - \sum_{i=1}^{s} c_i \eta(\lambda_i) < \varepsilon .$$

Proof. The positivity follows from concavity of η. To prove the other inequality let $A = \sum_i c_i \lambda_i$. By the Mean Value Theorem there is μ_i between λ_i and A such that

$$\lambda_i(\log \lambda_i - \log A) = \frac{\lambda_i}{\mu_i}\big(\lambda_i - A\big), \qquad \text{for } 0 < \lambda_i \leq 1 ,$$

and if $\lambda_i = 0$ the same is true. Thus we have

$$\begin{aligned} 0 &\leq \eta(A) - \sum c_i \eta(\lambda_i) \\ &= -\sum c_i \lambda_i (\log A - \log \lambda_i) \\ &= \sum_{\lambda_i \leq A} c_i \frac{\lambda_i}{\mu_i}\big(\lambda_i - A\big) + \sum_{\lambda_i > A} c_i \frac{\lambda_i}{\mu_i}\big(\lambda_i - A\big) \\ &< \sum_{\lambda_i > A} c_i \frac{\lambda_i}{A}\big(\lambda_i - A\big) \\ &\leq \frac{1}{A} \sum c_i \lambda_i \max |\lambda_i - A| \\ &= \varepsilon . \end{aligned}$$

QED.

Lemma 3.7. Let M be a von Neumann algebra with a faithful normal trace τ with $\tau(1) = 1$. Let $N \subset M$ be a finite dimensional von Neumann subalgebra containing 1. Let $x \in M_1^+$. Then

$$0 \le \eta\tau(x) - \tau\eta E_N(x) \le \left\| E_N^0(x) \right\|.$$

Proof. We can write

$$E_N(x) = \sum_{k=1}^{s} \lambda_k q_k \qquad \lambda_k \in [0,1],$$

where q_k is a minimal projection in N, and $\sum_{k=1}^{s} q_k = 1$. It follows that

$$\tau\eta(E_N(x)) = \sum_k \tau(q_k)\eta(\lambda_k)$$

$$\tau(x) = \tau(E_N(x)) = \sum_k \tau(q_k)\lambda_k .$$

Therefore we have by Lemma 3.6, since $\sum_k \tau(q_k) = 1$

$$\begin{aligned} 0 &\le \eta(\tau(x)) - \tau\eta(E_N(x)) \\ &= \eta\Big(\sum_k \tau(q_k)\lambda_k\Big) - \sum_k \tau(q_k)\eta(\lambda_k) \\ &\le \max_k |\lambda_k - \sum_j \tau(q_j)\lambda_j| \\ &= \|E_N(x) - \tau(x)\| . \end{aligned}$$

QED.

Proof of Theorem 3.3. Let $M \subset L(\mathbf{F}_\infty)$ be a finite dimensional von Neumann subalgebra. To show $H(\alpha) = 0$ we must show

$$\lim_{k\to\infty} \tfrac{1}{k} H(M, \alpha(M), \ldots, \alpha^{k-1}(M)) = 0 .$$

Since M is finite dimensional, given $\eta > 0$ there exists a finite subset S of the set G of generators of $\mathbf{F}_\infty$ such that $M \overset{\eta}{\subset} L(\mathbf{F}_S)$, i.e. for all $x \in M_1$ there is $y \in L(\mathbf{F}_S)_1$ such that $\|x - y\|_2 < \eta$. By [4, Thm. 1] if $\varepsilon > 0$ there is $\delta > 0$ such that if N is a von Neumann subalgebra of $L(\mathbf{F}_\infty)$ and $M \overset{\delta}{\subset} N$ then the relative entropy $H(M|N) < \varepsilon$. By [1, Thm. 4.2] if $\eta = \frac{1}{105}\delta$ (or use [4, Lem. 5]) then there is a finite dimensional von Neumann subalgebra

$N \subset L(\mathbf{F}_S)$ such that $M \overset{\delta}{\subset} N$, hence $H(M|N) < \varepsilon$. But then by property (F) of the function H in [4],

$$\begin{aligned}\tfrac{1}{k}H(M,\alpha(M),\ldots,\alpha^{k-1}(M)) &\leq \tfrac{1}{k}H(N,\alpha(N),\ldots,\alpha^{k-1}(N)) + H(M|N)\\ &< \tfrac{1}{k}H(N,\ldots,\alpha^{k-1}(N)) + \varepsilon\,.\end{aligned}$$

Thus it suffices to show $\frac{1}{k}H(N,\ldots,\alpha^{k-1}(N)) < \varepsilon$ for k sufficiently large and $\varepsilon > 0$ given.

Since S is a finite set and α is free on G, there is by Lemma 3.2 $p \in \mathbf{N}$ such that the sets $\alpha^{np}(S)$, $n \in \mathbf{Z}$, are all disjoint. Since the sequence $\left\{\frac{1}{k}H(N,\ldots,\alpha^{k-1}(N))\right\}_{k\in\mathbf{N}}$ converges it suffices by Lemma 3.4 to show

$$\tfrac{1}{k}H(N,\alpha^p(N),\ldots,\alpha^{p(k-1)}(N)) \leq \varepsilon \qquad \text{for large } k\,. \tag{4}$$

Let $\theta = \alpha^p$. If $N_j = \theta^{j-1}(N)$, $j = 1,2,\ldots$, they satisfy the assumptions of Lemma 2.5. Let $r = r(\varepsilon/2, N)$ of Lemma 2.5. Fix $k \in \mathbf{N}$ with $k > \frac{2r}{\varepsilon}H(N)$. Let $(x_{i_1\ldots i_k}) \in S_k$ be a partition of unity in $L(\mathbf{F}_\infty)^+$. We shall show that

$$\tfrac{1}{k}\Big(\sum_{i_1\ldots i_k}\eta\tau(x_{i_1\ldots i_k}) - \sum_{\ell=1}^{k}\sum_{i_\ell}\tau\eta E_{N_\ell}(x^\ell_{i_\ell})\Big) < \varepsilon\,,$$

which will show (4).

By Lemma 3.5 we have

$$\begin{aligned}&\sum_{i_1\ldots i_k}\eta\tau(x_{i_1\ldots i_k}) - \sum_{\ell=1}^{k}\sum_{i_\ell}\tau\eta E_{N_\ell}(x^\ell_{i_\ell}) \leq\\ &\quad\leq \Big(\sum_{i_1\ldots i_k}\eta\tau(x_{i_1\ldots i_k}) - \sum_{\ell}\sum_{i_\ell}\eta\tau(x^\ell_{i_\ell})\Big) + \sum_{\ell}\Big(\sum_{i_\ell}\eta\tau(x^\ell_{i_\ell}) - \tau\eta E_{N_\ell}(x^\ell_{i_\ell})\Big)\\ &\quad\leq \sum_{\ell=1}^{k}\Big|\sum_{i_\ell}\eta\tau(x^\ell_{i_\ell}) - \tau\eta E_{N_\ell}(x^\ell_{i_\ell})\Big|\,.\end{aligned} \tag{5}$$

By Lemma 2.5 there is a subset $J \subset \mathbf{N}$ with card $J \leq r$ such that

$$\sum_{i_1\ldots i_k}\left\|E^0_{N_\ell}(x_{i_1\ldots i_k})\right\| < \varepsilon/2\,, \qquad \ell \notin J\,.$$

It follows that for $\ell \notin J$

$$\begin{aligned}\sum_{i_\ell}\left\|E^0_{N_\ell}(x^\ell_{i_\ell})\right\| &= \sum_{i_\ell}\Big\|E^0_{N_\ell}\Big(\sum_{i_1\ldots i_{\ell-1},i_{\ell+1}\ldots i_k} x_{i_1\ldots i_k}\Big)\Big\|\\ &\leq \sum_{i_\ell}\sum_{i_1\ldots i_{\ell-1},i_{\ell+1}\ldots i_k}\left\|E^0_{N_\ell}(x_{i_1\ldots i_k})\right\|\\ &< \varepsilon/2\,.\end{aligned}$$

Thus by Lemma 3.7

$$\Big|\sum_{i_\ell}\eta\tau(x^\ell_{i_\ell})-\tau\eta(E_{N_\ell}(x^\ell_{i_\ell}))\Big|<\varepsilon/2\,,\qquad \ell\notin J\,.$$

We therefore have, using (5),

$$\begin{aligned}&\tfrac{1}{k}\Big[\sum_{i_1\ldots i_k}\eta\tau(x_{i_1\ldots i_k})-\sum_{\ell=1}^{k}\sum_{i_\ell}\tau\eta E_{N_\ell}(x^\ell_{i_\ell})\Big]\\&\qquad\le\tfrac{1}{k}\sum_{\ell\in J}\Big|\sum_{i_\ell}\eta\tau(x^\ell_{i_\ell})-\tau\eta E_{N_\ell}(x^\ell_{i_\ell})\Big|+\tfrac{\varepsilon}{2k}(k-\operatorname{card}J)\,.\end{aligned}\tag{6}$$

Since

$$\Big|\sum_{i_\ell}\eta\tau(x^\ell_{i_\ell})-\tau\eta E_{N_\ell}(x^\ell_{i_\ell})\Big|\le H(N_\ell)=H(N)$$

we obtain that the expression in (6) is smaller than

$$\tfrac{1}{k}\operatorname{card}J\,H(N)+\varepsilon/2\,,$$

which is smaller than ε since $k>\frac{2r}{\varepsilon}H(N)$. Thus independently of the partition $(x_{i_1\ldots i_k})$ we have

$$\tfrac{1}{k}\Big(\sum_{i_1\ldots i_k}\eta\tau(x_{i_1\ldots i_k})-\sum_{\ell}\sum_{i_\ell}\tau\eta E_{N_\ell}(x^\ell_{i_\ell})\Big)<\varepsilon$$

for $k>\frac{2r}{\varepsilon}H(N)$, proving the theorem.

QED.

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